Math 206A Lecture 15 Notes

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1 Simplex Methods and Klee-Minty Cubes

1.1 Simplex methods

Let $P \subseteq \mathbb{R}^d$ be a convex polytope and $\Gamma = \Gamma(P) = (V, E)$ be a graph. Suppose $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a linear function non constant on edges of P. We want to find $v \in V$ such that φ is maximized at v. The idea is to start at some $s \in V$, and walk along the graph edges in increasing direction with respect to φ. How do we know which edge to take if there are multiple increasing edges?

Definition 1.1. A **pivot rule** is a method of determining how to choose which up edges to walk along.

Example 1.1. We can choose the edge with steepest ascent, max value of φ at the endpoint, lexicographically first, random, or a different pivot rule.

In reality, people use a pivot rule different from all of these.

1.2 Klee-Minty Cubes

Theorem 1.1 (Klee-Minty,1972)**.** *There exists a simple $P \subseteq \mathbb{R}^d$ with $|V| = 2^d$, $f_{d-1} = 2d$, and $\alpha(P) \cong \alpha(C_d)$. such that the length of the simplex method can be 2^{d-1}.*

We will prove a weaker theorem, which does not rely on a rule. The construction is the same, and you can modify it to work with a given pivot rule.

Theorem 1.2. *There exists such a $P \subseteq \mathbb{R}^d$ with a maximum increasing path of length $2^d - 1$.*

Proof. First is a sketch of the intuitive idea. Proceed by induction on d. We can do this for $d = 2$, by creating an isosceles trapezoid. Given the construction for d, place a small copy of the construction parallel to the construction, and connect them with edges to get the construction for $d + 1$.

Explicitly, let

$$x_1 \leq 5$$
$$4x_1 + x_2 \leq 5^2$$
$$8x_1 + 4x_2 + x_3 \leq 5^3$$
$$\vdots$$
$$2^d x_1 + 2^{d-1} x_2 + \cdots + 4x_{d-1} + x_d \leq 5^d$$

and $x_i \geq 0$ for $i = 1, \ldots, d$. Let $\varphi = 2^{d-1}x_1 + 2^{d-2}x_2 + \cdots + 2x_{d-1} + x_d$. The vertices are when we have exactly d equalities.

The function φ has a minimum at $(0, \ldots, 0)$ and a maximum at $(0, \ldots, 0, 5^d)$. Why? Let's construct a Hamiltonian path.

vertex	φ
$v_0 = (0, \ldots, 0)$	0
$v_1 = (5, 0, \ldots, 0)$	$5 \cdot 2^{d-1}$
$v_2 = (5, 5, 0, \ldots, 0)$	$5 \cdot 2^{d-1} + 5 \cdot 2^{d-2}$
$v_3 = (0, 25, 0, \ldots, 0)$	$25 \cdot 2^{d-2}$
$v_4 = (0, 25, 25, 0, \ldots, 0)$	
$\vdots$	

The idea is that we have an iterative procedure for finding the path, and we can prove that it works by induction. The Klee-Minty cube is designed to have the bounds grow exponentially faster than the Morse function φ. $\square$

The moral of the story is that simplex methods can be exponentially slow. But in practice, people still use them. One reason is that they use randomized pivot rules. Moreover, since computers only calculate things up to finite precision, you eventually don't even see the extra paths after a certain point.

Theorem 1.3 (Spielman-Teng). *For "random" constraints, the simplex method runs in polynomial time.*